# Peregrine: Toward Fastest FALCON Based on GPV Framework

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Young-Sik Kim — Chosun University
Eun-young Seo — Chosun University
Joon-Woo Lee — Chung-ang University
Jong-Seon No — Seoul National University

# Lattice Based Signatures



- **FALCON**
  - Selected as a standard signature by NIST in July 2022
  - Hash-and-Sign Signature
  - GPV framework + NTRU lattices + fast Fourier sampling

| Algorithms | Category | Security | Public Key | Private Key | Signature Size |
|---|---|---|---|---|---|
| **Dilithium** | MLWE | 128 | 1,312B | 2,528B | 2,420B |
| | | 256 | 2,592B | 4,864B | 4,595B |
| **Falcon** | NTRU | 128 | **897B** | **1,281B** | **666B** |
| | | 256 | **1,793B** | **2,305B** | **1,330B** |
| **SPHINCS+** | Hash | 128 | 32B | 64B | 17,088B |
| | | 256 | 64B | 128B | 49,856B |

# GPV Framework

- Gentry, Peikert and Vaikuntanathan (GPV) framework
  - For construction hash-and-sign lattice-based signature scheme

**Public Key**

A full-rank matrix $A \in Z_q^{n \times m} (m > n)$ generating a $q$-ary lattice $L_q = L(A^t)$

**Private Key**

A trapdoor matrix $B \in Z_q^{m \times m}$ generating the lattice $L_q^{\perp} = L(B)$, orthogonal to $L_q$, such as $B \times A^t = 0$

**Signature**

a short value $\vec{s} \in Z_q^m$ such that $\vec{s} \cdot A^t = H(m)$

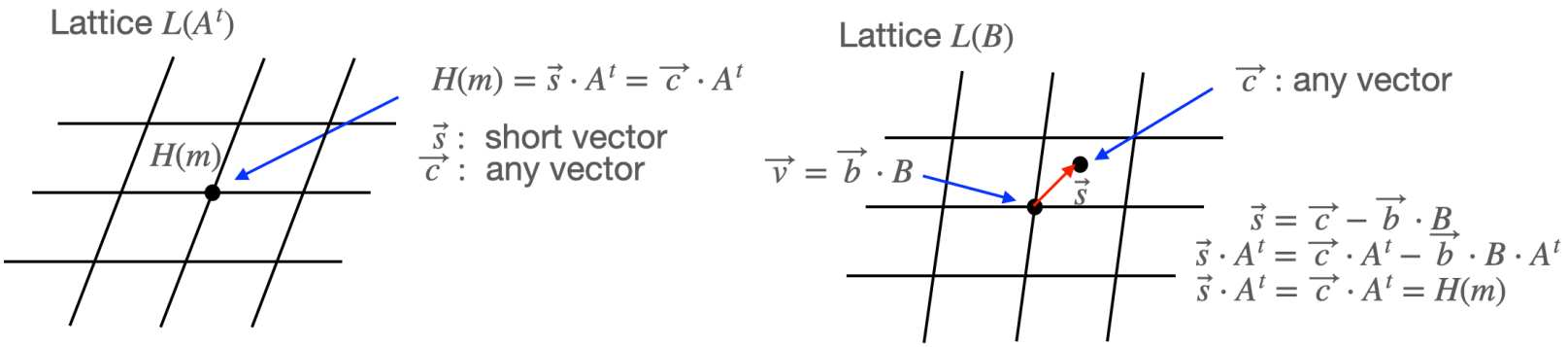
A message $\boldsymbol{m}$

a hash function $H: \{0,1\}^* \rightarrow Z_q^n$

a random salt $r$, $H(m||r)$

# How to find the short $s$

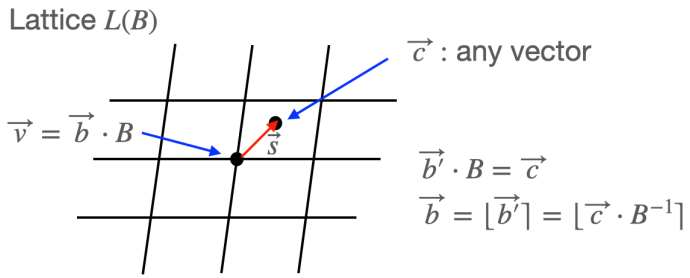
- Find any $\vec{c}$, satisfying $\vec{c} \cdot A^t = H(m)$
- Find the lattice point $\vec{v} \in L_q^{\perp}$, close to $\vec{c}$
- Then, $\vec{s}A^t = (\vec{c} - \vec{v})A^t = \vec{c}A^t - \vec{b}BA^t = H(m)$

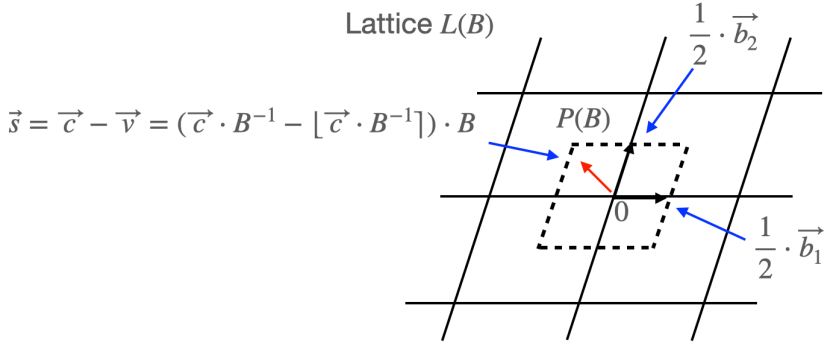


# Finding short $s$

## Babai's round-off algorithm

- Lattice $L(B) = \{ \sum_{i=1}^{m} x_i \cdot \overrightarrow{b_i} \,|\, x_i \in Z \}$, row vectors of $B = \{\overrightarrow{b_1}, \overrightarrow{b_2}, \cdots, \overrightarrow{b_m}\}$
- Half-open Parallelipiped $P(B) = \{ \sum_{i=1}^{m} x_i \cdot \overrightarrow{b_i} \,|\, -\frac{1}{2} \le x_i < \frac{1}{2} \}$
- Step 1 : find $\overrightarrow{b'}$ satisfying $\overrightarrow{b'} \cdot B = \overrightarrow{c}$ such as $\overrightarrow{b'} = \overrightarrow{c} \cdot B^{-1}$
- Step 2 : $\overrightarrow{b} = \lfloor \overrightarrow{b'} \rceil = \lfloor \overrightarrow{c} \cdot B^{-1} \rceil$
- Step 3 : $\overrightarrow{v} = \overrightarrow{b} \cdot B = \lfloor \overrightarrow{c} \cdot B^{-1} \rceil \cdot B$
- Short vector : $\vec{s} = \overrightarrow{c} - \overrightarrow{v} = (\overrightarrow{c} \cdot B^{-1} - \lfloor \overrightarrow{c} \cdot B^{-1} \rceil) \cdot B \in P(B)$
- Parallel operation possible, but longer vector $\vec{s}$ than the Babai's nearest plane algorithm



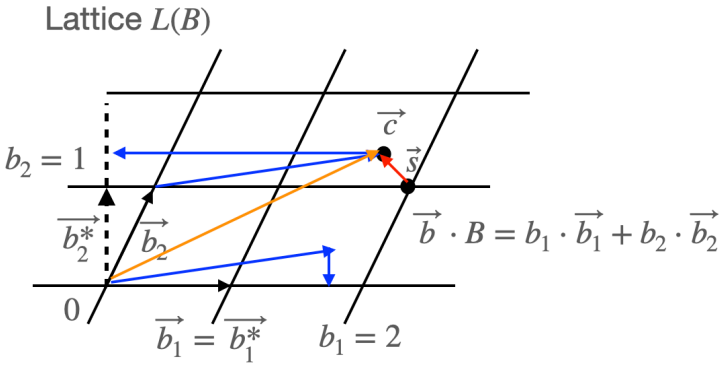



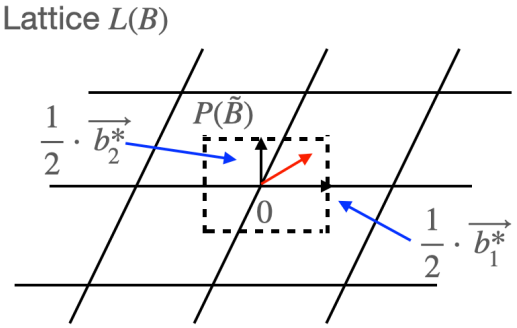
# Finding short $s$

## Babai's nearest plane algorithm

- $\tilde{B}$ : Gram-Schmidt orthogonalization of $B$
- $\tilde{B} = L \cdot B$, row vectors of $\tilde{B} = \{\overrightarrow{b_1^*}, \overrightarrow{b_2^*}, \cdots, \overrightarrow{b_m^*}\}$, $\overrightarrow{b_i^*} = \overrightarrow{b_i} - \sum_{j=1}^{i-1} \frac{< \overrightarrow{b_i}, \overrightarrow{b_j^*} >}{< \overrightarrow{b_j^*}, \overrightarrow{b_j^*} >} \overrightarrow{b_j^*}$, $i > 1$, $\overrightarrow{b_1^*} = \overrightarrow{b_1}$
- Half-open Parallelipiped $P(\tilde{B}) = \{ \sum_{i=1}^{m} x_i \cdot \overrightarrow{b_i^*} \mid -\frac{1}{2} \le x_i < \frac{1}{2} \}$
- Step 0 : $i = m$
- Step 1 : $b_i = \lfloor \frac{< \overrightarrow{c}, \overrightarrow{b_i^*} >}{< \overrightarrow{b_i^*}, \overrightarrow{b_i^*} >} \rceil$, $\overrightarrow{b} = (b_1, b_2, \cdots, b_m)$
- Step 2 : $\overrightarrow{c} = \overrightarrow{c} - b_i \cdot \overrightarrow{b_i}$
- Step 3 : $i = i - 1$, if $i \ge 1$ go to Step 1
- Sequential operation, but shorter $\vec{s}$ than the Babai's round-off algorithm

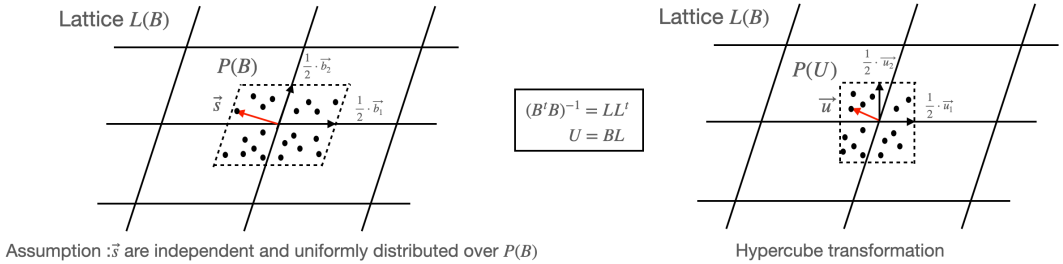
$$\overrightarrow{c} - \overrightarrow{b} \cdot B \in P(\tilde{B})$$



# Problem of deterministic algorithms

- Solving Hidden Parallelepiped Problem(Learing a Parallelepiped:Cryptanalysis of GGH and NTRU Signatures)
- **Input** : A polynomial number of samples uniformly distributed over $P(B)$
- **Output** : Approximation of rows of $B = \{\overrightarrow{b_1}, \overrightarrow{b_2}, \cdots, \overrightarrow{b_m}\}$
- 1. **Gram Leakage :** $\vec{s} = \overrightarrow{x}B$, $\overrightarrow{x} = (x_1, x_2, \cdots, x_m)$, $x_i$ has uniform distribution over $[-\frac{1}{2}, \frac{1}{2}] \rightarrow E[\vec{s}^t\vec{s}] = B^tB/12$
- 2. **Hypercube Transformation :** $G = B^tB$ is the symmetric positive definite matrix. There exists the unique lower-triangular matrix $L$, such that $G^{-1} = LL^t$. Then, $U = BL$ is an orthogonal matrix and $P(U)$ is a unit hypercube.
- 3. **Learning** $U$ : Define a function $m_{\overrightarrow{u},k}(\overrightarrow{w}) = E[< \overrightarrow{u}, \overrightarrow{w} >^k]$, $\overrightarrow{u}$ is uniformly distributed over $P(U)$, $\overrightarrow{u} = \overrightarrow{x}U = \sum_{i=1}^{m} x_i\overrightarrow{u_i}$, $\overrightarrow{w} \in R^n$. When $k = 4$, $m_{\overrightarrow{u},4}(\overrightarrow{w}) = \frac{||\overrightarrow{w}||^4}{3} - \frac{2}{15}\sum_{i=1}^{m} < \overrightarrow{u_i}, \overrightarrow{w} >^4$ has the global minimum $\frac{1}{5}$ and the minimum is obtained at $\pm\overrightarrow{u_1}, \cdots, \pm\overrightarrow{u_m}$ over the unit sphere of $R^n$. Gradient descent algorithm can be used to find the minimum point.
- 4. **Approximation of** $B$ : $B = UL^{-1}$
- $B$ and $\tilde{B}$ can be revealed from uniform samples over $P(B)$ and $P(\tilde{B})$ .



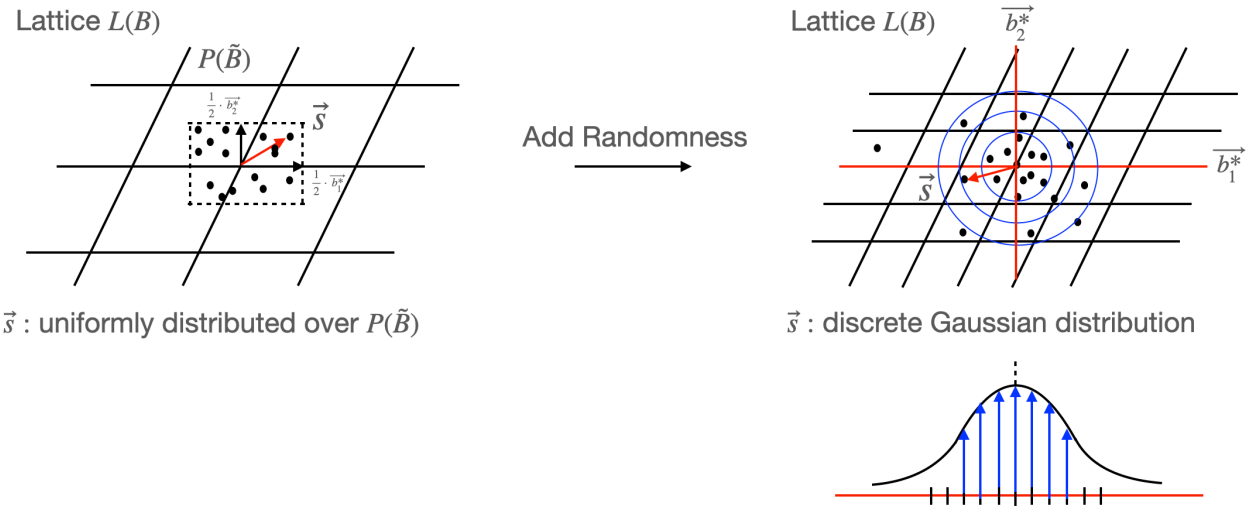

Assumption :$\vec{s}$ are independent and uniformly distributed over $P(B)$

Hypercube transformation

# GPV Framework to FALCON

## Add randomness with discrete Gaussian R.V.

- Hide the private key from addtion of randomness
- $m-$ dimensional Gaussian function $\rho : R^n \to (0,1]$ is defined as $\rho(\vec{x}) = e^{-\pi \cdot <\vec{x},\vec{x}>}$, and $\rho_B(\vec{x}) = \rho(B^{-1}\vec{x}) = \rho_{\sqrt{\Sigma}}(\vec{x})$, $BB^t = \Sigma$
- Discrete Gaussian distribution over a lattice $L$ : for all $\vec{x} \in L + \vec{c}$, $D_{L+\vec{c},\sqrt{\Sigma}} = \dfrac{\rho_{\sqrt{\Sigma}}(\vec{x})}{\rho_{\sqrt{\Sigma}}(L+\vec{c})}$
- Babai's nearest plane algorithm + discrete Gaussian r.v.



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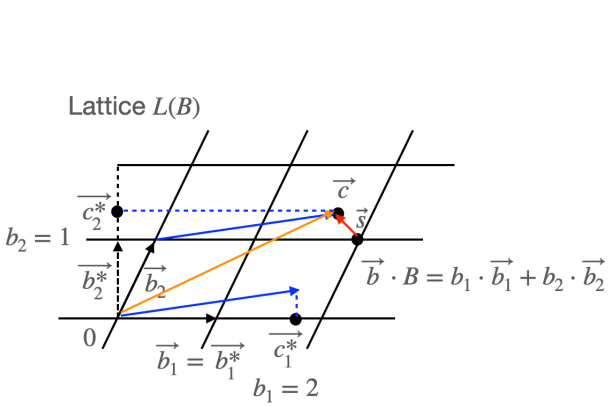
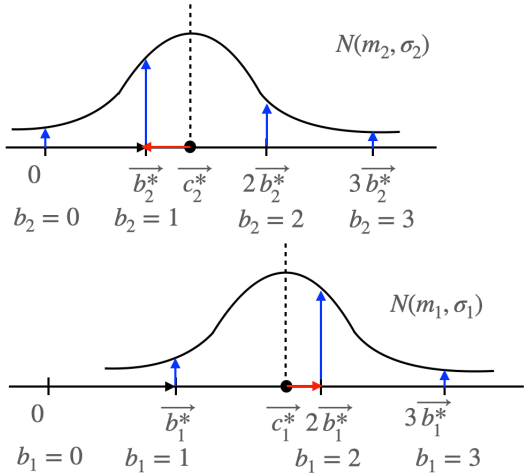
  Babai's nearest plane algorithm + discrete Gaussian r.v.




Babai's nearest plane algorithm

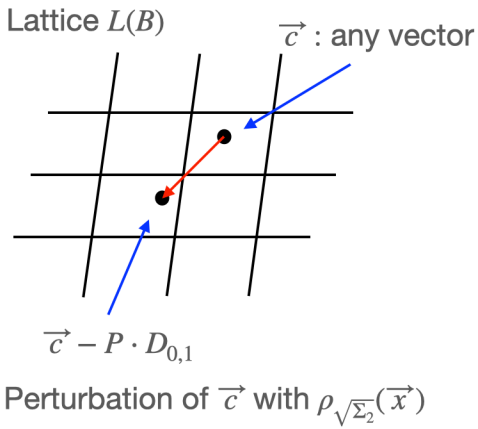



Randomized sampling

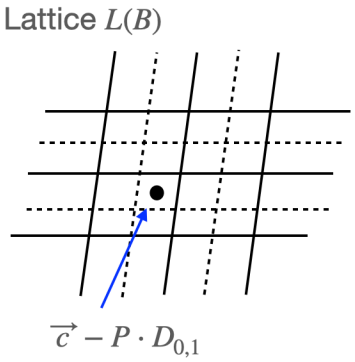
# Peikert's algorithm

## Add randomness with discrete Gaussian R.V.

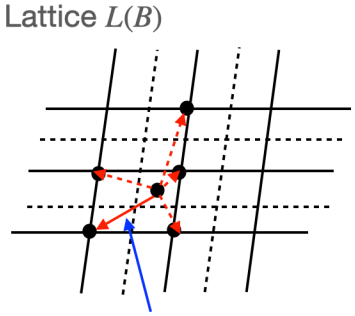
- Babai's rounding-off algorithm + discrete Gaussian r.v. → skewed(elliptical) Gaussian : the skew mirrors the geometry of the basis
- Perturbation(offline precomputation) + Babai's round-off algorithm + discrete Gaussian r.v.
- Main concept : summation of two discrete Gaussian r.v. → discrete Gaussian r.v.
- Define : matrix $P$, its covariance matrix $\Sigma_2 = PP^t$, private key matrix $B$, its covariance matrix $\Sigma_1 = r^2 \cdot BB^t$
- Offline phase : find the matrix $P$ such as $\Sigma_1 + \Sigma_2 = s^2 I$
- Online phase : sample $\vec{x} \leftarrow P \cdot D_{0,1}$ and move $\vec{c}$ to $\vec{c} - \vec{x}$ and apply Babai's round-off algorithm + discrete Gaussian randomization such as $\lfloor B^{-1}(\vec{c} - \vec{x}) \rceil_r$ , where $\lfloor v \rceil_r$ has distribution $v + D_{Z-v,r}$




Perturbation of $\vec{c}$ with $\rho_{\sqrt{\Sigma_2}}(\vec{x})$




Babai's round-off algorithm



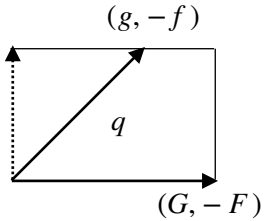

randomize with $\rho_{\sqrt{\Sigma_1}}(\vec{x})$

# FALCON : NTRU Lattices

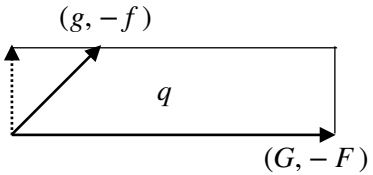
| | |
|---|---|
| Polynomial ring | $Z_q[x]/\phi(x)$, where $\phi(x) = x^n + 1$ for $n = 2^k$ |
| NTRU Equation | $fG - gF = q \bmod \phi(x) f, g, F, G \in Z_q[x]/\phi(x)$ |
| Public Key | $A = (1 \;\; h)$, where $h = f^{-1} \cdot g \bmod q$ |
| Private Key | $B = \begin{pmatrix} g & -f \\ G & -F \end{pmatrix} \rightarrow \det B = q \rightarrow \sqrt{q} \leq \|\|B\|\|_{GS}$ |

- $f$ and $g$ are generated randomly by discrete Gaussian distribution $D_{z,\sigma}$ with $\sigma = 1.17\sqrt{q/2n}$
- $F$ and $G$ are calculated by solving the NTRU equation
- Signature norm is proportional to the Gram-Schmidt norm of $B$, $\|\|B\|\|_{GS} = \max_{\tilde{b}_i \in \tilde{B}} \|\|\tilde{B}_i\|\|$
- $\|\|B\|\|_{GS}$ is minimized for $\|\|(f, g)\|\| \approx 1.17\sqrt{q}$ (experimental result)
- Discard $f$ and $g$ if $\|\|B\|\|_{GS} \geq 1.17\sqrt{q}$

$$B_{2n\times 2n} = \left(\begin{array}{cc|cc} g_0, & g_1, \cdots, & g_{n-1}, & -f_0, -f_1, \cdots, -f_{n-1} \\ -g_{n-1}, & g_0, \cdots, g_{n-2}, & & f_{n-1}, -f_1, \cdots, -f_{n-2} \\ \multicolumn{2}{c|}{\vdots} & & \vdots \\ -g_1, & -g_2, \cdots, g_0, & & f_1, f_2, \cdots, -f_0 \\ \hline G_0, & G_1, \cdots, G_{n-1}, & & -F_0, -F_1, \cdots, -F_{n-1} \\ -G_{n-1}, & G_0, \cdots, G_{n-2}, & & F_{n-1}, -F_1, \cdots, -F_{n-2} \\ \multicolumn{2}{c|}{\vdots} & & \vdots \\ -G_1, & -G_2, \cdots, G_0, & & F_1, F_2, \cdots, -F_0 \end{array}\right)$$



Good Basis



Bad Basis

# FALCON : Fast Fourier Sampling

How to find the lattice point $v$ close to $c$

FALCON

Babai's nearest plane + Klein's algorithm (discrete Gaussian randomization) in FFT domain

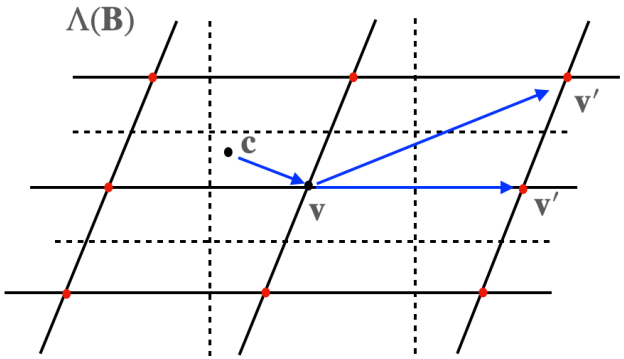
Sequential sampling

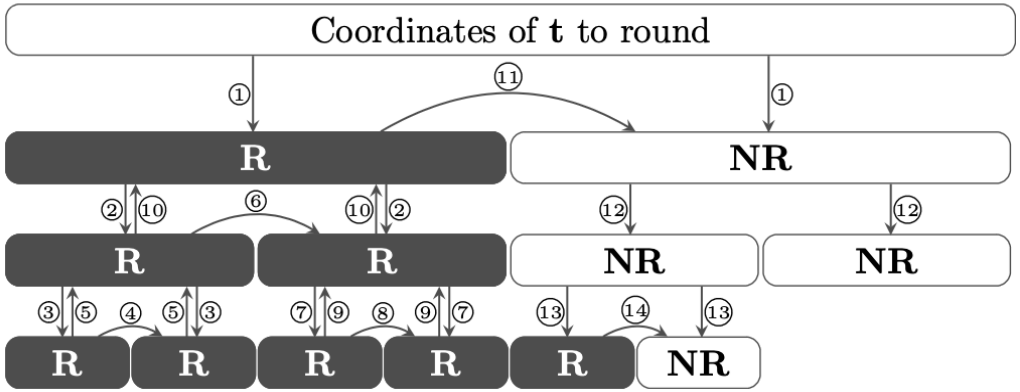
$$(t_0, t_1) \cdot B - (z_0\ z_1) \cdot B = P(\tilde{B})$$

Parallelepiped of orthogonal basis $\tilde{B}$

$z_1 : [t_1]$ + Gaussian random variable

$z_0 : [t_0 + (t_1 - z_1) \cdot l]$ + Gaussian random variable



# FALCON : Verification

- **Verification**
  - Check that the received signature is short enough vector

**Algorithm 7** verify($M$, **sig**, **pk**, $\lfloor \beta^2 \rfloor$)

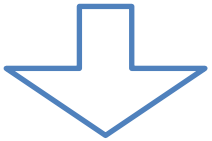
**Require:** A message $M$, a signature $\mathbf{sig} = (r, s_1)$, a public key $\mathbf{pk} = h \in \mathbb{Z}_q[x]/\phi$, a bound $\lfloor \beta^2 \rfloor$

**Ensure:** Accept or reject

1: $c \leftarrow H(M||r)$
2: $s_1 \leftarrow c - s_2 \cdot h \mod (\phi, q)$
3: **if** $||(s_1, s_2)|| \leq \lfloor \beta^2 \rfloor$ **then**
4: accept
5: **else**
6: reject
7: **end if**

# Complexity of FALCON

- Secure implementation of discrete Gaussian r.v.
- Floating point operation in the FFT and discrete Gaussian r.v.
- Recursive sequential sampling



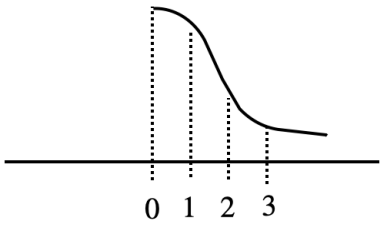
- To overcome sequentail sampling : hybrid Peikert's algorithm and Falcon→MITAKA
- We tried in Peregrine : **Babai's round-off algorithm** + **centered Binomial r.v.** + NTT only in the signature process

# Peregrine

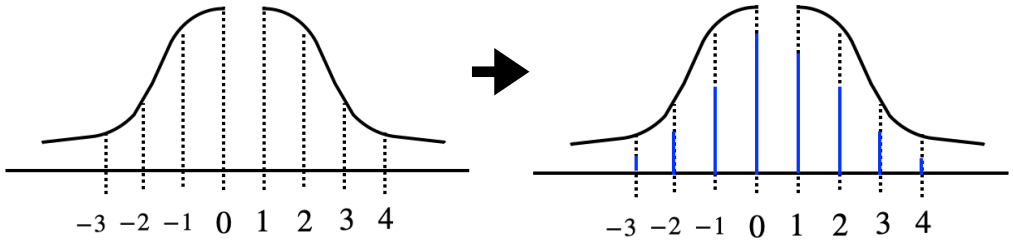
- **Discrete Gaussian v.s. centered Binomial r.v.**
  - Implementation of the centered binomial random variable is much simpler than the discrete Gaussian random variable and can cope with the side channel attack.

discrete Gaussian r.v.

BaseSampler



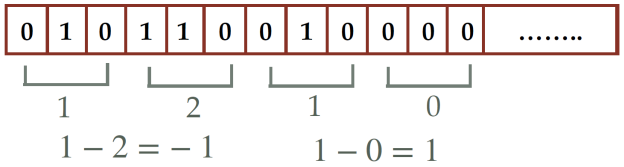

SamplerZ




centered Binomial r.v.

Pseudo random number generation

Counting # of 1 outputs




```
for(i = 0; i<10; i++)
{
    r = get_rng_u64(rng);
    printf("%164u",r);
    d = 0;
    for (j = 0; j < 3; j++)
      d += (r >> j) & 0x249249;

    a[0] = d & 0x7;
    b[0] = (d >> 3) & 0x7;
    a[1] = (d >> 6) & 0x7;
    b[1] = (d >> 9) & 0x7;
    a[2] = (d >> 12) & 0x7;
    b[2] = (d >> 15) & 0x7;
    a[3] = (d >> 18) & 0x7;
    b[3] = (d >> 21);

    s[4 * i + 0] = (int16_t)(a[0] - b[0]);
    s[4 * i + 1] = (int16_t)(a[1] - b[1]);
    s[4 * i + 2] = (int16_t)(a[2] - b[2]);
    s[4 * i + 3] = (uint16_t)(a[3] - b[3]);
}
```

# Peregrine

- **Key Generation**

Public Key

$$h(x) = f^{-1}(x) \cdot g(x) \in Z_q[x]/\phi(x)$$

Random polynomials $f(x), g(x) \in Z_q[x]/\phi(x)$ generated by random variable with centered binomial distribution

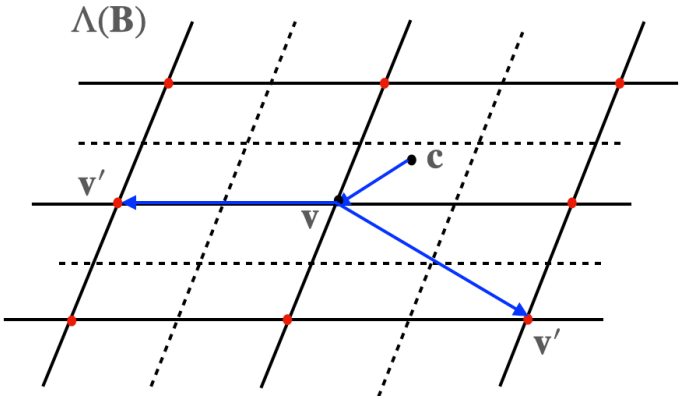
$$P[X = x] = \frac{\mu!}{((\mu/2)+x)! \cdot ((\mu/2)-x)!} \cdot 2^{-\mu},$$
$$-\mu/2 \leq x \leq \mu/2$$

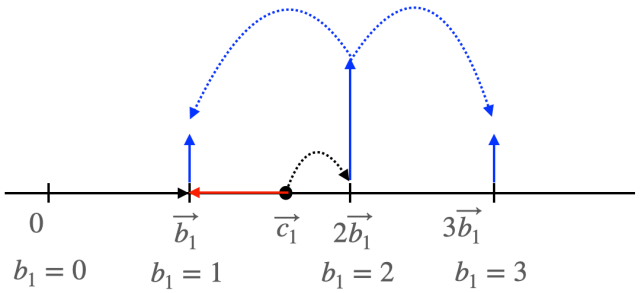
- The centered Binomial distribution with $\mu$=24 or 26 is very similar with the discrete Gaussian distribution which is used for random public key generation in FALCON.
- Simulation result in Peregrine shows that the proposed centered binomial distribution generates the public key and secret key, which are derived by solving the NTRU equation, successfully in similar running time with FALCON.

# Peregrine

- **Signature**

| | |
|---|---|
| Based on the GPV framework | Lattice-based hash-and-sign signature |
| Babai's round-off algorithm + randomization | Integer operation only without floating point FFT operation |
| RNS + NTT | ModDown algorithm is used to implement the round-off algorithm |
| Randomization | Centered binomial distribution with $\mu = 6$ is used to insert the randomness |




Randomize with centered Binomial r.v.

# Peregrine

- **Signature**

| | |
|---|---|
| Message Poly. | $c = H(M \mid\mid r) \in Z_q[x]/\phi(x)$ |
| Signature | Short vector $\mathbf{s} = (s_1, s_2) \in (Z_q[x]/\phi(x))^2$ satisfying $\mathbf{s} \cdot \mathbf{A}^{\mathbf{t}} = c$, where $\mathbf{A} = (1, h)$ |

- Need to find a polynomial vector $\mathbf{t} = \left(t_1, t_2\right) \in \left(Q[x]/\phi\right)^2$ satisfying $\mathbf{t} \cdot \mathbf{B} = \mathbf{c} = (c, 0)$
- $\mathbf{t} = \left(t_1, t_2\right) = \left(\frac{c \cdot f}{q}, -\frac{c \cdot F}{q}\right) = (R_1 + I_1,\ R_2 + I_2)$, where $R_l = \sum_{i=0}^{n-1} r_{lj} \cdot x^j$,

$l = 1, 2,\ -0.5 < r_{lj} \le 0.5$ and $I_i = \sum_{j=0}^{n-1} i_{lj} \cdot x^j,\ l = 1, 2,\ i_{lj} \in Z.$

Signature Generation

$$s = (t - z') \cdot B = \left(c - (I_1 + J_1) \cdot g - \left(I_2 + J_2\right) \cdot G, \left(I_1 + J_1\right) \cdot f + \left(I_2 + J_2\right) \cdot F\right) mod(\phi, q)$$

# Problems and Future Works

- Randomization with centered Binomial distribution is not enough to protect the private key information $B$.
- In order for FALCON to be used on various platforms, further research on reducing the implementation complexity of FALCON is needed.

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# Thank You!